数理解析研究所講究録
第 1841 巻 2013 年 42-49

A geometric constant induced by the Dunkl-Williams inequality

新潟大学大学院・自然科学研究科 水口 洋康 (Hiroyasu Mizuguchi)
Department of Mathematical Science,
Graduate School of Science and Technology, Niigata University

新潟大学・理学部 斎藤 吉助 (Kichi-Suke Saito)
Department of Mathematics, Faculty of Science, Niigata University

新潟大学大学院・自然科学研究科 田中 亮太朗 (Ryotaro Tanaka)
Department of Mathematical Science,
Graduate School of Science and Technology, Niigata University

1 Introduction

In this note, we mainly consider about the Dunkl-Williams constant. In particular, we describe some recent results obtained in [19].

Throughout this note, the term "normed linear space" always means a real normed linear space which has two or more dimension. For a normed linear space X, let B_X and S_X denote the unit ball and the unit sphere of X, respectively. In 1964, Dunkl and Williams [7] showed the following simple inequalities: Let X be a normed linear space. Then the inequality

$$\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|\leq\frac{4\|x-y\|}{\|x\|+\|y\|}$$

holds for all $x, y\in X\setminus\{0\}$, and if X is an inner product space, the stronger inequality

$$\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|\leq\frac{2\|x-y\|}{\|x\|+\|y\|}$$

holds for all $x, y\in X\backslash\{0\}$. These inequalities are so called the Dunkl-Williams inequality. In the same paper, it was also proved that for any $\varepsilon>0$ there exist $x, y\in(\mathbb{R}^{2}, \|\cdot\|_{1})$ such that

$$\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|>(4-\varepsilon)\frac{\|x-y\|}{\|x\|+\|y\|}.$$

This means that the constant 4 is the best possible choice for the Dunkl-Williams inequality in the space $(\mathbb{R}^{2}, \|\cdot\|_{1})$. There are many result related to this inequality (cf. [1, 4, 5, 6, 16, 17, 20, 21, 22, 23, 24], and so on).

2 The Dunkl-Williams inequality

In this section, we list some results related to the Dunkl-Williams inequality. First, we see the original proof of the inequality.

Theorem 2.1 (The Dunkl-Williams inequality). *Let X be a normed linear space. Then, the inequality*

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{4\|x-y\|}{\|x\|+\|y\|}$$

holds for all $x, y \in X \setminus \{0\}$, and if X is an inner product space, the stronger inequality

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2\|x-y\|}{\|x\|+\|y\|}$$

holds for all $x, y \in X \setminus \{0\}$.

Proof. Let x and y be two nonzero elements of X. Then we have

$$\begin{aligned} \|x\| \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| &\leq \|x\| \left\| \frac{x}{\|x\|} - \frac{y}{\|x\|} \right\| + \|x\| \left\| \frac{y}{\|x\|} - \frac{y}{\|y\|} \right\| \\ &= \|x-y\| + \big| \|x\| - \|y\| \big| \\ &\leq 2\|x-y\|. \end{aligned}$$

Replacing x with y, we also have

$$\|y\| \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq 2\|x-y\|.$$

Therefore we obtain

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{4\|x-y\|}{\|x\|+\|y\|}.$$

Next, we assume that X is an inner product space. Then, for each nonzero elements $x, y \in X$, we obtain

$$\begin{aligned} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2 &= 2 - 2\mathrm{Re} \left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|} \right\rangle \\ &= \frac{1}{\|x\|\|y\|}(2\|x\|\|y\| - 2\mathrm{Re}\langle x, y \rangle) \\ &= \frac{1}{\|x\|\|y\|}\left(\|x-y\|^2 - (\|x\| - \|y\|)^2 \right). \end{aligned}$$

Hence we have

$$\begin{aligned} &\|x-y\|^2 - \left(\frac{\|x\|+\|y\|}{2} \right)^2 \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2 \\ &= \frac{(\|x\| - \|y\|)^2}{\|x\|\|y\|}\left((\|x\|+\|y\|)^2 - \|x-y\|^2 \right) \geq 0, \end{aligned}$$

and so the inequality

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2\|x-y\|}{\|x\|+\|y\|}$$

holds. $\square$

Dunkl and Williams asked in their paper [7] whether the second inequality in Theorem 2.1 characterizes inner product spaces. A bit later, Kirk and Smiley [14] solved this problem affirmatively. They used the following result of Lorch [15].

Lemma 2.2 (Lorch, 1948). *Let X be a normed linear space. Then, X is an inner product space if and only if $x, y \in X$ and $\|x\| = \|y\|$ implies $\|\alpha x + \alpha^{-1}y\| \geq \|x+y\|$ for all $\alpha > 0$.*

Now, we show the result of Kirk and Smiley.

Theorem 2.3 (Kirk-Smiley, 1964). *Let X be a normed linear space. Then, X is an inner product space if the inequality*

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2\|x-y\|}{\|x\| + \|y\|}$$

holds for all $x, y \in X \setminus \{0\}$.

Proof. Let x and y be nonzero elements of X such that $\|x\| = \|y\|$, and let $\alpha > 0$. Applying the inequality for αx and $\alpha^{-1}y$, we have

$$\begin{aligned} \|\alpha x + \alpha^{-1}y\| &\geq \frac{\|\alpha x\| + \|\alpha^{-1}y\|}{2} \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \\ &= \frac{\alpha + \alpha^{-1}}{2} \|x+y\| \\ &\geq \|x+y\|. \end{aligned}$$

Thus, X is an inner product space by Lemma 2.2. □

As a consequence of Theorems 2.1 and 2.3, it turns out that a normed linear space X is an inner product space if and only if the inequality

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2\|x-y\|}{\|x\| + \|y\|}$$

holds for all $x, y \in X \setminus \{0\}$. Thus, the best possible choice for the Dunkl-Williams inequality measures "how much" the space is close (or far) to be an inner product space. Motivated by this fact, Jiménez-Melado et al. [13] defined the Dunkl-Williams constant $DW(X)$ of a normed linear space X as the best constant for the Dunkl-Williams inequality, that is,

$$DW(X) = \sup \left\{ \frac{\|x\| + \|y\|}{\|x-y\|} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| : x, y \in X \setminus \{0\},\ x \neq y \right\}.$$

As was mentioned in Section 1, $DW((\mathbb{R}^2, \|\cdot\|_1)) = 4$, and Theorems 2.1 and 2.3 are restated as follows: Let X be a normed linear space. Then,

(i) $2 \leq DW(X) \leq 4$.

(ii) X is an inner product space if and only if $DW(X) = 2$.

Furthermore, it is known that $DW(X) = 4$ if and only if the space X is not uniformly non-square. Recall that a normed linear space X is said to be uniformly non-square if there exists $\delta > 0$ such that $x, y \in S_X$ and $\|x-y\| > 2(1-\delta)$ implies $\|x+y\| \leq 2(1-\delta)$. However, the Dunkl-Williams constant is very hard to calculate. In fact, except the case of $DW(X) = 2$ or 4, there have been probably no other example of the space X for which $DW(X)$ is determined precisely.

3 A calculation method for $DW(X)$

In [19], we constructed a new calculation method for the Dunkl-Williams constant. In this section, we describe the calculation method. As an application, we determine the precise value of $DW(\ell_2\text{-}\ell_\infty)$, where $\ell_2\text{-}\ell_\infty$ is the Day-James space defined as the space $\mathbb{R}^2$ endowed with the norm $\|\cdot\|_{2,\infty}$ given by

$$\|(a,b)\|_{2,\infty} = \begin{cases} \|(a,b)\|_2 & \text{if } ab \geq 0, \\ \|(a,b)\|_\infty & \text{if } ab \leq 0. \end{cases}$$

for all $(a,b) \in \mathbb{R}^2$.

When constructing a method, the notion of Birkhoff orthogonality plays an important role. We recall that for two elements x, y of a normed linear space X, x is said to be Birkhoff orthogonal to y, denoted by $x \perp_B y$, if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{R}$. Obviously, Birkhoff orthogonality is always homogeneous, that is, $x \perp_B y$ implies $\alpha x \perp_B \beta y$ for all $\alpha, \beta \in \mathbb{R}$. More details about Birkhoff orthogonality can be found in Birkhoff [3], Day [8, 9] and James [10, 11, 12].

To construct a calculation method, we introduce some notations. Suppose that X is a normed linear space. For each $x \in S_X$, let $V(x)$ be a subset of X defined by $V(x) = \{y \in X : x \perp_B y\}$. For each $x \in S_X$ and each $y \in V(x)$, we define $\Gamma(x,y)$ and $m(x,y)$ by

$$\Gamma(x,y) = \left\{ \frac{\lambda + \mu}{2} : \lambda \leq 0 \leq \mu,\ \|x + \lambda y\| = \|x + \mu y\| \right\}$$

and $m(x,y) = \sup\{\|x + \gamma y\| : \gamma \in \Gamma(x,y)\}$, respectively. Furthermore, let

$$M(x) = \sup\{m(x,y) : y \in V(x)\}.$$

Using these notions, we obtain a new calculation method for the Dunkl-Williams constant.

Theorem 3.1 ([19]). *Let X be a normed linear space. Then,*

$$DW(X) = 2\sup\{M(x) : x \in S_X\}.$$

If $\dim X = 2$, we have the following improvement of the preceding theorem.

Theorem 3.2 ([19]). *Let X be a normed linear space with $\dim X = 2$. Then,*

$$DW(X) = 2\sup\{M(x) : x \in \mathrm{ext}(B_X)\},$$

where $\mathrm{ext}(B_X)$ denotes the set of all extreme points of B_X.

When we put this theory into practice, the following results are needed.

Proposition 3.3. *Let X be a normed linear space. Suppose that $x \in S_X$ and $y \in V(x)$. Then, the following hold:*

(i) $0 \in V(x)$.

(ii) $\alpha y \in V(x)$ *for all* $\alpha \in \mathbb{R}$.

(iii) $m(x,0) = 1 \leq m(x,y)$.

(iv) $m(x, \alpha y) = m(x, y)$ *for all* $\alpha \in \mathbb{R} \setminus \{0\}$.

Proposition 3.4. *Let* X, Y *be normed linear spaces and let* $x \in S_X$ *and* $y \in V(x)$. *Suppose that* T *is an isometric isomorphism from* X *onto* Y. *Then, the following hold:*

(i) $m(Tx, Ty) = m(x, y)$.

(ii) $M(Tx) = M(x)$.

Proposition 3.5. *Let* X *be a normed linear space and let* $x \in S_X$ *and* $y \in V(x) \setminus \{0\}$. *Then,* $\Gamma(x, y)$ *is a bounded subset of* $\mathbb{R}$. *Furthermore,* $m(x, y) = \max\{\|x+\alpha y\|, \|x+\beta y\|\}$, *where* $\alpha = \inf \Gamma(x, y)$ *and* $\beta = \sup \Gamma(x, y)$, *respectively.*

Theorem 3.6. *Let* X *be a normed linear space and let* $x \in S_X$ *and* $y \in V(x)$. *Suppose that* $\{x_n\}$ *is a sequence in* S_X *which converges to* x. *If the sequence* $\{y_n\}$ *satisfies* $y_n \in V(x_n)$ *for each* $n \in \mathbb{N}$ *and converges to* y, *then*

$$m(x, y) \leq \liminf_{n \to \infty} m(x_n, y_n).$$

All of these results can be found in [19].

4 The Dunkl-Williams constant of the space ℓ_2-ℓ_∞

Applying Theorem 3.2, we obtain the following example.

Theorem 4.1 ([19]). $DW(\ell_2\text{-}\ell_\infty) = 2\sqrt{2}$.

To prove Theorem 4.1, we need a lot of works. First, one can easily show that

$$\mathrm{ext}(B_{\ell_2\text{-}\ell_\infty}) = \{(a, b) \in \mathbb{R}^2 : ab \geq 0,\ a^2 + b^2 = 1\} \cup \{(1, -1), (-1, 1)\}.$$

Now, let $M_0 = \sup\{M((a, b)) : 0 < b < a,\ a^2 + b^2 = 1\}$. Then, we have the following lemma by Theorems 3.2 and 3.6, and Proposition 3.4.

Lemma 4.2. $DW(\ell_2\text{-}\ell_\infty) = 2\max\{M_0, M((1, -1))\}$.

We remark that $0 < b < a$ and $a^2 + b^2 = 1$ implies $b < 1/\sqrt{2} < a$. Next, to calculate $M(x)$, we find the set $V(x)$ for each x.

Lemma 4.3. *Suppose that* $0 < b < a$ *and* $a^2 + b^2 = 1$. *Then,*

$$V((a, b)) = \{\alpha(b, -a) \in \mathbb{R}^2 : \alpha \in \mathbb{R}\}.$$

Lemma 4.4. $V((1, -1)) = \{(a, b) \in \mathbb{R}^2 : ab \geq 0\}$.

To reduce the amount of computation, we make use of Proposition 3.3.

Lemma 4.5. *Suppose that* $0 < b < a$ *and* $a^2 + b^2 = 1$. *Then,*

$$M((a, b)) = m((a, b), (b, -a)).$$

Lemma 4.6. $M((1, -1)) = \sup\{m((1, -1), (a, b)) : 0 < b < a,\ a^2 + b^2 = 1\}$.

We need to determine the set $\Gamma(x,y)$ to calculate the value of $m(x,y)$.

Lemma 4.7. *Suppose that* $0<b<a$ *and* $a^2+b^2=1$. *Then,*

$$\Gamma((a,b),(b,-a)) = \begin{cases} [0, b/a] & \text{if } a \le 2b, \\ [0, (a+b-\sqrt{2ab})/(a-b)] & \text{if } a > 2b. \end{cases}$$

Lemma 4.8. *Suppose that* $0<b<a$ *and* $a^2+b^2=1$. *Then,*

$$\Gamma((1,-1),(a,b)) = [b-a, 0].$$

Now we prove Theorem 4.1. Proposition 3.5 is used in this phase.

Proof of Theorem 4.1. Suppose that $0<b<a$ and $a^2+b^2=1$. First, we assume that $a \le 2b$. Then, by Proposition 3.5 and Lemma 4.7, we have

$$\begin{aligned} M((a,b)) &= m((a,b),(b,-a)) \\ &= \max\left\{ \|(a,b)\|_{2,\infty}, \left\|(a,b)+\frac{b}{a}(b,-a)\right\|_{2,\infty} \right\} \\ &= \frac{1}{a}. \end{aligned}$$

On the other hand, if $0<b<a$ and $a^2+b^2=1$, then $a\le 2b$ if and only if $a \le 2/\sqrt{5}$. Hence we obtain

$$\begin{aligned} \{M(a,b) : 0<b<a\le 2b,\ a^2+b^2=1\} &= \{1/a : 1/\sqrt{2} < a \le 2/\sqrt{5}\} \\ &= [\sqrt{5}/2, \sqrt{2}). \end{aligned}$$

Next, we suppose that $a>2b$. Then we have

$$0 < \frac{a+b-\sqrt{2ab}}{a-b} < 1.$$

Since the function $t \mapsto \|(a,b)+t(b,-a)\|$ is convex and increasing on $[0,\infty)$, we obtain

$$\begin{aligned} M((a,b)) &= m((a,b),(b,-a)) \\ &= \left\|(a,b)+\frac{a+b-\sqrt{2ab}}{a-b}(b,-a)\right\|_{2,\infty} \\ &\le \|(a,b)+(b,-a)\|_{2,\infty} \\ &= \|(a+b,b-a)\|_{2,\infty} \\ &= a+b \le \sqrt{2}(a^2+b^2)^{1/2} = \sqrt{2} \end{aligned}$$

by Proposition 3.5 and Lemma 4.7. Thus, we have

$$M_0 = \sup\{M((a,b)) : 0<b<a,\ a^2+b^2=1\} = \sqrt{2}.$$

Finally, by Proposition 3.5 and Lemma 4.8, we obtain

$$\begin{aligned} m((1,-1),(a,b)) &= \|(1,-1)+(b-a)(a,b)\|_{2,\infty} \\ &= \|(a^2+b^2,-a^2-b^2)+(ab-a^2,b^2-ab)\|_{2,\infty} \\ &= \|(ab+b^2,-a^2-ab)\|_{2,\infty} \\ &= (a+b)\|(b,-a)\|_{2,\infty} \\ &= a(a+b) \le \sqrt{2}. \end{aligned}$$

This implies that $M((1,-1)) \le \sqrt{2} = M_0$.

Thus, by Lemma 4.2, we have

$$DW(\ell_2\text{-}\ell_\infty) = 2\max\{M_0, M((1,-1))\} = 2M_0 = 2\sqrt{2}. \quad \square$$

References

[1] A. M. Al-Rashed, Norm inequalities and characterization of inner product spaces, J. Math. Anal. Appl., 176(1993), 587-593.

[2] M. Baronti and P. L. Papini, Up and down along rays, Riv. Mat. Univ. Parma, 2*(1999), 171–189.

[3] G. Birkoff, Orthogonality in linear metric spaces, Duke Math. J., 1(1935), 169–172.

[4] F. Dadipour, M. Fujii and M. S. Moslehian, Dunkl-Williams inequality for operators associated with p-angular distance, Nihonkai Math. J., 21(2010), 11-20.

[5] F. Dadipour and M. S. Moslehian, An approach to operator Dunkl-Williams inequalities, Publ. Math. Debrecen, 79(2011), 109–118.

[6] F. Dadipour and M. S. Moslehian, A characterization of inner product spaces related to the p-angular distance, J. Math. Anal. Appl., 371(2010), 667–681.

[7] C. F. Dunkl and K. S. Williams, A simple norm inequality, Amer. Math. Monthly, 71(1964), 53–54.

[8] M. M. Day, Polygons circumscribed about closed convex curves, Trans. Amer. Math. Soc., 62(1947), 315–319.

[9] M. M. Day, Some characterizations of inner product spaces, Trans. Amer. Math. Soc., 62(1947), 320–337.

[10] R. C. James, Orthogonality in normed linear spaces, Duke Math. J., 12(1945), 291–302.

[11] R. C. James, Inner products in normed linear spaces, Bull. Amer. Math. Soc., 53(1947), 559–566.

[12] R. C. James, Orthogonality and linear functionals in normed linear spaces, Trans. Amer. Math. Soc., 61(1947), 265–292.

[13] A. Jiménez-Melado, E. Llorens-Fuster and E. M. Mazcunan-Navarro, The Dunkl-Williams constant, convexity, smoothness and normal structure, J. Math. Anal. Appl., 342(2008), 298–310.

[14] W. A. Kirk and M. F. Smiley, Another characterization of inner product spaces, Amer. Math. Monthly, 71(1964), 890–891.

[15] E. R. Lorch, On certain implications which characterize Hilbert space, Ann. of Math. (2), 49 (1948), 523–532.

[16] L. Maligranda, Simple norm inequalities, Amer. Math. Monthly, 113(2006), 256-260.

[17] J. L. Massera and J. J. Schäffer, Linear differnetial equations and functional analysis I, Ann. of Math., 67(1958), 517-573.

[18] R. E. Megginson, An Introduction to Banach Space Theory, Springer-Verlag, New York, 1998.

[19] H. Mizuguchi, K. -S. Saito and R. Tanaka, On the calculation of the Dunkl-Williams constant of normed linear spaces, to appear in Cent. Eur. J. Math.

[20] M. S. Moslehian and F. Dadipour, Characterization of equality in a generalized Dunkl-Williams inequality, J. Math. Anal. Appl., 384(2011), 204–210.

[21] M. S. Moslehian, F. Dadipour, R. Rajić and A. Marić, A glimpse at the Dunkl-Williams inequality, Banach J. Math. Anal., 5(2011), 138–151.

[22] J. E. Pečarić and R. Rajić, Inequalities of the Dunkl-Williams type for absolute value operators, J. Math. Inequal., 4(2010), 1–10.

[23] K. -S. Saito and M. Tominaga, A Dunkl-Williams type inequality for absolute value operators, Linear Algebra Appl., 432(2010), 3258–3264.

[24] K. -S. Saito and M. Tominaga, A Dunkl-Williams inequality and the generalized operator version, International Series of Numerical Mathematics, 161(2012), 137-148, Birkhauser.